Conjunctive Spaces

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Abstract

We introduce the concept of a conjunctive space. Every reflexive, symmetric relation induces a conjunctive space. This leads to a general definition of "point" via an induced partial order.

Keywords. conjunctive space, partial order, chain, point, nest.

1 Introduction

Euclid [1] defined a point as "that which has no part." One could equivalently define a point x in a set as something with the property that, if it is a member of sets y and z, then y and z meet each other. This inspires a more general definition of a point: If we have defined the notion of "contact" between two entities, we can immediately define the notion of "point" as follows: x is a point if and only if whenever x contacts y and z, then y and z contact each other. Clearly "contact" is a relation; in this section we denote it by the binary operator $\circ$. We stipulate some intuitively reasonable properties of $\circ$.

Reflexivity Every entity contacts itself: $x \circ x$

Symmetry Contact is mutual: $x \circ y \Leftrightarrow y \circ x$

Equivalence An entity is uniquely determined by its contacts: $\forall z\,(x \circ z \leftrightarrow y \circ z) \Rightarrow x = y$

We call any such relation a conjunctive relation. We may now define a point:

$$x \text{ is a point} \Leftrightarrow \forall y, z\,((x \circ y \text{ and } x \circ z) \to y \circ z)\,. \tag{1.1}$$

In the next section will begin with a more precise definition of a conjunctive relation, and develop its properties.

2 Conjunctive Relation

Let X be a nonempty set.

Definition We say that a relation $\circ$ on X is conjunctive if and only if

(1) $\forall x \in X\,(x \circ x)$,

(2) $\forall x_1, x_2 \in X\,(x_1 \circ x_2 \leftrightarrow x_2 \circ x_1)$, and

(3) $\forall y \in X\,(x_1 \circ y \leftrightarrow x_2 \circ y)$ implies $x_1 = x_2$.

Notation We call a nonempty set X together with a conjunctive relation $\circ$, a conjunctive space $(X, \circ)$.

Remark Any reflexive and symmetric relation induces a conjunctive space. Indeed, let R be such a relation on a set Y. Define X to be the collection of equivalence classes under the equivalence relation satisfying $y_1 \sim y_2$ if and only if $\forall z \in Y\,(y_1 R z \leftrightarrow y_2 R z)$. For any equivalence classes $\bar{y}_1, \bar{y}_2$ in X, we define $\bar{y}_1 \circ \bar{y}_2$ if and only if $y_1 R y_2$. It is easily seen that $(X, \circ)$ is a conjunctive space.

Definition For $x \in X$, we define the function $\alpha : X \to 2^X$ satisfying

$$\alpha(x) = \{y \in X; x \circ y\}. \tag{2.1}$$

Note that α has the following properties:

1. $\forall x \in X, x \in \alpha(x)$,
2. $\forall x, y \in X, x \in \alpha(y)$ and $y \in \alpha(x)$, and
3. α is injective.

Definition For $a, b \in X$, we define the join and meet of a and b as follows. We say that $x = a \vee b$ if and only if $\forall z \in X\,(x \circ z \leftrightarrow a \circ z$ or $b \circ z\}$. We say that $x = a \wedge b$ if and only if $\forall z \in X\,(x \circ z \leftrightarrow a \circ z$ and $b \circ z\}$.

Examples

(1) Let X be some nonempty set, and define the relation $x_1 \circ x_2$ if and only if $x_1 = x_2$.

(2) Let X be a collection of nonempty subsets of some set Y. For any $x_1, x_2 \in X$, define the relation $x_1 \circ x_2$ if and only if $x_1 \cap x_2 \neq \emptyset$. An interesting instance of this example is obtained when we take

$X = \{\{p \in \mathbb{Q}; p_1 \leqslant p \leqslant p_2\}; p_1, p_2 \in \mathbb{Q}, p_1 \leqslant p_2\}$. We call this the rational conjunctive space and will discuss it in greater detail below.

(3) Let X be the collection of collections of nested nonempty subsets of some nonempty set Y. For $N_1, N_2 \in X$, define the relation $N_1 \circ N_2$ if and only if $\forall n_1 \in N_1 \forall n_2 \in N_2 (n_1 \cap n_2 \neq \emptyset)$.

(4) Let $\mathcal{L} = \mathcal{L}(A, \Omega, Z, I)$ be a propositional calculus with the usual operators $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, 0, 1\} \subseteq \Omega$. Let Σ be the collection of sentences of $\mathcal{L}$. For finite $S \subseteq \Sigma$, define the operator Ξ such that $\Xi(S) \equiv \bigwedge_{x \in S} x$, with $\bigwedge_{x \in \emptyset} x \equiv 1$. Let $X = \{\emptyset \subset S \subseteq \Sigma; S \text{ is finite}, \Xi(S) \neq 0\}$. Define an equivalence relation on members of X such that $S_1 \sim S_2$ if and only if for all finite $T \subseteq \Sigma, \Xi(S_1 \cup T) \neq 0 \leftrightarrow \Xi(S_2 \cup T) \neq 0$. Consider the collection of equivalence classes $C = \{\bar{S}; S \in X\}$. For $S_1, S_2 \in X$, define the relation $\bar{S_1} \circ \bar{S_2}$ if and only if $\Xi(S_1 \cup S_2) \neq 0$.

(5) Let Σ be a nonempty collection of subsets of a set X, and let $a : X \rightarrow 2^X$ be a function such that $x \in a(x)$ and $x \in a(y) \leftrightarrow y \in a(x)$. Define a relation R on subsets of X such that $C_1 R C_2$ if and only if

$$\forall C_3 \left(C_1 \subseteq \bigcap_{x_3 \in C_3} a(x_3) \Leftrightarrow C_2 \subseteq \bigcap_{x_3 \in C_3} a(x_3) \right).$$

(6) Consider the equivalence relation on groups satisfying $G_1 \sim G_2$ if and only if S is a simple subgroup of $G_1 \leftrightarrow S$ is a simple subgroup of G_2. On the equivalence classes we then define the relation $\circ$ satisfying $\bar{G}_1 \circ \bar{G}_2 \leftrightarrow$ there exists a simple group S (depending on G_1 and G_2) that is a subgroup of both G_1 and G_2. As defined, $\circ$ is a conjunctive relation.

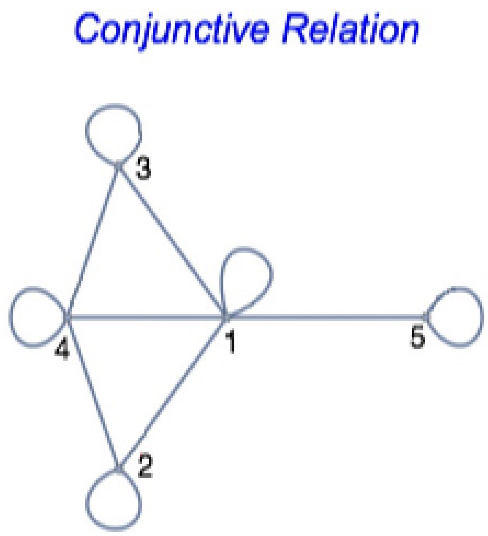

Figure 1: An example of a conjunctive relation.

3 Cartesian Products of Conjunctive Spaces

Given two conjunctive spaces $(X, \circ)$ and $(Y, \bullet)$, we may define the product space $(X, \circ) \times (Y, \bullet)$ as the conjunctive space $(X \times Y, \star)$ satisfying

$$(x_1, y_1) \star (x_2, y_2) \Leftrightarrow x_1 \circ x_2 \text{ and } y_1 \bullet y_2. \tag{3.1}$$

It is straightforward to verify the reflexive and symmetric properties of this definition. To show the equivalence property, suppose for some $(x_1, y_1) \neq (x_2, y_2)$ we have $\forall (p, q) \in (X, Y) \left((x_1, y_1) \star (p, q) \leftrightarrow (x_2, y_2) \star (p, q)\right)$.

Case 1: $x_1 \neq x_2$. This means

$\forall (p, q) \in (X, Y) \left(x_1 \circ p \text{ and } y_1 \bullet q \leftrightarrow x_2 \circ p \text{ and } y_2 \bullet q\right)$, which implies

$\forall p \in X \left(x_1 \circ p \text{ and } y_1 \bullet y_1 \leftrightarrow x_2 \circ p \text{ and } y_2 \bullet y_1\right)$. Since we always have

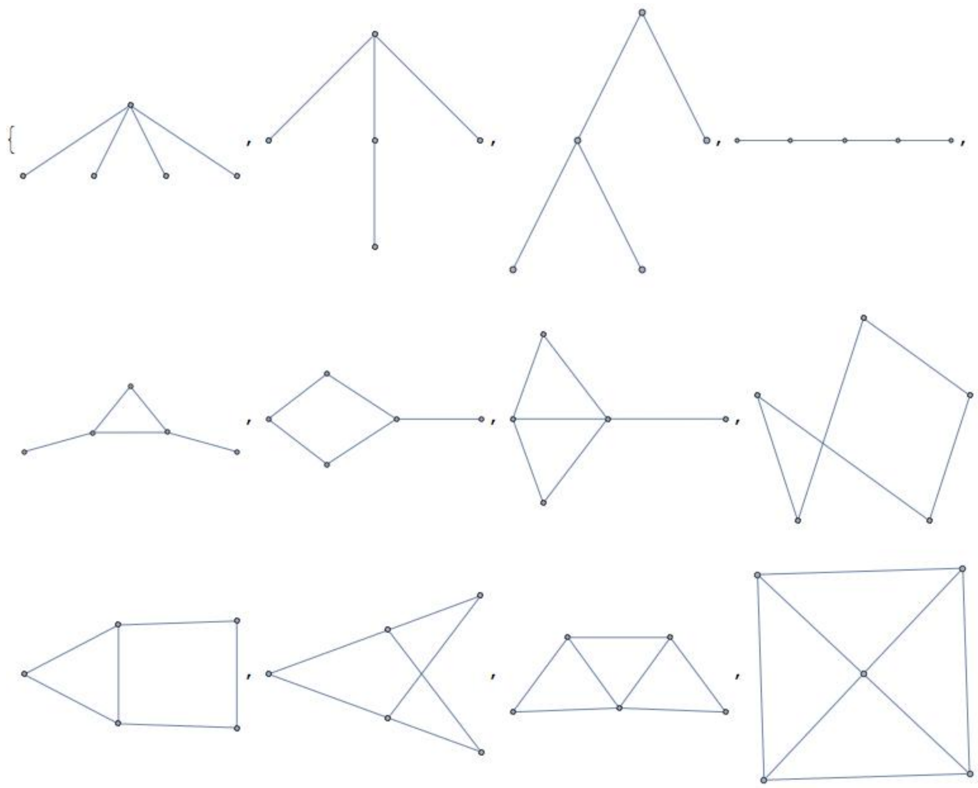

Figure 2: Connected Conjunctive Spaces On Five Points

$y_1 \bullet y_1$, this means $\forall p \in X\ (x_1 \circ p \leftrightarrow x_2 \circ p$ and $y_2 \bullet y_1)$. Since by assumption $x_1 \neq x_2$, we can find p_1 such that $x_1 \circ p_1$ is true and for which $x_2 \circ p_1$ is false. Since $x_1 \circ p_1$, and $x_1 \circ p_1 \leftrightarrow x_2 \circ p_1$ and $y_2 \bullet y_1$, we get $x_2 \circ p_1$ for a contradiction.

Case 2: $y_1 \neq y_2$. A contradiction is obtained with a similar argument. Thus we have $(x_1, y_1) = (x_2, y_2)$. ■

Proposition 3.1 *The product space* $(X \times Y, \star) \equiv (X, \circ) \times (Y, \bullet)$ *has an induced partial order* $\prec_\star$ *satisfying*

$$(x_1, y_1) \prec_\star (x_2, y_2) \Leftrightarrow x_1 \prec_\circ x_2 \text{ and } y_1 \prec_\bullet y_2 \tag{3.2}$$

Proof $(\Rightarrow)$ Suppose $(x_1, y_1) \prec_\star (x_2, y_2)$. Fix p, q such that $p \circ x_1$ and $q \bullet y_1$. Then $(p, q) \star (x_1, y_1)$. Since $(x_1, y_1) \prec_\star (x_2, y_2)$, we have $(p, q) \star (x_2, y_2)$, which implies $p \circ x_2$ and $q \bullet y_2$, whence $x_1 \prec_\circ x_2$ and $y_1 \prec_\bullet y_2$.

$(\Leftarrow)$ Suppose $x_1 \prec_\circ x_2$ and $y_1 \prec_\bullet y_2$. Fix (p, q) such that $(p, q) \star (x_1, y_1)$. Then $p \circ x_1$ and $q \bullet y_1$. Since $x_1 \prec_\circ x_2$ and $y_1 \prec_\bullet y_2$, we get $p \circ x_2$ and $q \bullet y_2$, whence $(p, q) \star (x_2, y_2)$. This implies $(x_1, y_1) \prec_\star (x_2, y_2)$. ■

Proposition 3.2 *The points of the product space* $(X \times Y, \star) \equiv (X, \circ) \times (Y, \bullet)$ *are the members* $(p_1, p_2) \in (X, Y)$ *such that* p_1 *is a point in* $(X, \circ)$, *and* p_2 *is a point in* $(Y, \bullet)$.

Proof $(\Rightarrow)$ Suppose (p_1, p_2) is a point in $(X \times Y, \star)$. Fix $x_1 \in X$ and $y_1 \in Y$ such that $p_1 \circ x_1$ and $p_2 \bullet y_1$. Then $(p_1, p_2) \star (x_1, y_1)$. Since (p_1, p_2) is a point, we have $(p_1, p_2) \prec_\star (x_1, y_1)$, which implies $p_1 \prec_\circ x_1$ and $p_2 \prec_\bullet x_2$. These imply p_1 is a point in $(X, \circ)$, and p_2 is a point in $(Y, \bullet)$.

($\Leftarrow$) Suppose p_1 is a point in $(X, \circ)$, and p_2 is a point in $(Y, \bullet)$. Fix (x_1, y_1) such that $(p_1, p_2) \star (x_1, y_1)$. Then $p_1 \circ x_1$ and $p_2 \bullet y_1$. Since p_1 and p_2 are points, we get $p_1 \prec_\circ x_1$ and $p_2 \prec_\bullet y_1$. This implies $(p_1, p_2) \prec_\star (x_1, y_1)$, so (p_1, p_2) is a point in $(X \times Y, \star)$. $\blacksquare$

4 A Conjunctive Relation Induces a Partial Order

Let $(X, \circ)$ be a conjunctive space.

Definition Define the order $\prec$ satisfying $x_1 \prec x_2$ if and only if

$\forall y \in X\, (x_1 \circ y \to x_2 \circ y)$.

Proposition 4.1 *$\prec$ is a partial order on X.*

Proof (1) Let $x \in X$. Since clearly $\forall y \in X\, (x \circ y \to x \circ y)$, it follows that $x \prec x$. (2) Suppose $x_1 \prec x_2$ and $x_2 \prec x_1$. Fix $y \in X$ such that $x_1 \circ y$. Since $x_1 \prec x_2$, we have $x_2 \circ y$, thus $\forall y \in X\, (x_1 \circ y \leftrightarrow x_2 \circ y)$, which by property 3 of $\circ$ implies $x_1 = x_2$. (3) Suppose $x_1 \prec x_2$ and $x_2 \prec x_3$. Fix $y \in X$. Then $x_1 \circ y \to x_2 \circ y$ and $x_2 \circ y \to x_3 \circ y$, implying $x_1 \circ y \to x_3 \circ y$, thus $\forall y \in X\, (x_1 \circ y \to x_3 \circ y)$. But this just means $x_1 \prec x_3$. $\blacksquare$

Proposition 4.2 *If $x_1 \prec x_2$, then $x_1 \circ x_3$ implies $x_2 \circ x_3$.*

Proof Suppose $x_1 \prec x_2$ and $x_1 \circ x_3$. Since $x_1 \prec x_2$, we have $\forall y \in X\, (x_1 \circ y \to x_2 \circ y)$, so in particular $x_1 \circ x_3 \to x_2 \circ x_3$, and thus we get $x_2 \circ x_3$. $\blacksquare$

Corollary 4.3 *If $x_1 \prec x_2$, then $x_1 \circ x_2$.*

Proof Suppose $x_1 \prec x_2$. Then since $x_1 \circ x_1$, Proposition 4.2 immediately implies $x_2 \circ x_1$. This in turn implies $x_1 \circ x_2$. ■

Notation We write $\Phi(X, \circ)$ to denote the collection of chains of $(X, \circ)$ under the induced partial order $\prec$.

Corollary 4.4 *If $C \in \Phi(X, \circ)$, then $\forall x_1, x_2 \in C\,(x_1 \circ x_2)$.*

Proof Suppose C is a chain and $x_1, x_2 \in C$. Then $x_1 \prec x_2$ or $x_2 \prec x_1$. In either case, from Corollary 4.3 we get $x_1 \circ x_2$, hence $\forall x_1, x_2 \in C\,(x_1 \circ x_2)$, as desired. ■

Proposition 4.5 *Define a conjunctive relation $\circ$ on the set Y of nonempty proper subsets of some nonempty set X, such that $x_1 \circ x_2 \leftrightarrow (x_1 \cap x_2 \neq \emptyset)$. Then $x_1 \prec x_2 \Leftrightarrow x_1 \subseteq x_2$.*

Proof ($\Rightarrow$) Suppose $x_1 \prec x_2$. Then $\forall y \in Y\,(x_1 \circ y \to x_2 \circ y)$, i.e. $\forall y \in Y\,(x_1 \cap y \neq \emptyset \to x_2 \cap y \neq \emptyset)$. Since the set $x_2^c \in Y$, we have

$x_1 \cap x_2^c \neq \emptyset \to x_2 \cap x_2^c \neq \emptyset$. If $x_1 \cap x_2^c \neq \emptyset$, this implication would give $x_2 \cap x_2^c \neq \emptyset$, which is clearly false, hence $x_1 \cap x_2^c = \emptyset$. But this just means $x_1 \subseteq x_2$.

($\Leftarrow$) Suppose $x_1 \subseteq x_2$. Fix any $y \in Y$ such that $x_1 \cap y \neq \emptyset$. Since $x_1 \subseteq x_2$, we have $x_1 \cap y \subseteq x_2 \cap y$ and thus $x_2 \cap y \neq \emptyset$, hence $\forall y \in Y\,(x_1 \cap y \neq \emptyset \to x_2 \cap y \neq \emptyset)$. But this just means $x_1 \prec x_2$. ■

Remark In Example 1, $x_1 \prec x_2$ if and only if $x_1 = x_2$.

In Example 2, $x_1 \prec x_2$ if and only if $x_1 \subseteq x_2$.

In Example 3, $N_1 \prec N_2$ if and only if $\forall n_2 \in N_2 \exists n_1 \in N_1 (n_1 \subseteq n_2)$. Indeed, suppose $N_1 \prec N_2$, and for a contradiction suppose $\exists n_2 \in N_2 \forall n_1 \in N_1 (n_1 \setminus n_2 \neq \emptyset)$. By Corollary 4.3, we get $N_1 \circ N_2$. Fix $n_2 \in N_2$ such that $\forall n_1 \in N_1 (n_1 \setminus n_2 \neq \emptyset)$. Take $N_3 = \{n_1 \setminus n_2; n_1 \in N_1\}$. This is a chain of nonempty members. Clearly $\forall n_1 \in N_1 \forall n_3 \in N_3 (n_1 \cap n_3 \neq \emptyset)$, so since $N_1 \prec N_2$, we have $\forall n_2 \in N_2 \forall n_3 \in N_3 (n_2 \cap n_3) \neq \emptyset$, i.e. $\forall n_2 \in N_2 \forall n_3 \in \{n_1 \setminus n_2; n_1 \in N_1\} (n_2 \cap n_3) \neq \emptyset$, which is a clear contradiction, thus $\forall n_2 \in N_2 \exists n_1 \in N_1 (n_1 \subseteq n_2)$.

Conversely, suppose $\forall n_2 \in N_2 \exists n_1 \in N_1 (n_1 \subseteq n_2)$. Fix $n_2 \in N_2$. Fix N_3 such that $N_1 \circ N_3$. Then $\forall n_1 \in N_1 \forall n_3 \in N_3 (n_1 \cap n_3) \neq \emptyset$. Fix $n_3 \in N_3$. By hypothesis, we get $\exists n_1 \in N_1 (n_1 \subseteq n_2)$. Fix $n_1 \in N_1$ such that $n_1 \subseteq n_2$. Since $\forall n_1 \in N_1 \forall n_3 \in N_3 (n_1 \cap n_3) \neq \emptyset$, we get $n_1 \cap n_3 \neq \emptyset$. Since $n_1 \subseteq n_2$, we get $n_2 \cap n_3 \neq \emptyset$. Thus $\forall n_2 \in N_2 \forall n_3 \in N_3 (n_2 \cap n_3) \neq \emptyset$, which means $N_2 \circ N_3$. This gives $\forall N_3 (N_1 \circ N_3 \to N_2 \circ N_3)$, which means $N_1 \prec N_2$.

In Example 4, if $S_1 \cap \mathrm{Ann}(\Xi(S_2)) \neq \emptyset$, then $\bar{S}_1 \prec \bar{S}_2$ implies $S_1 \subseteq S_2$. Here $\mathrm{Ann}(y) \equiv \{z \in \Sigma; z \neq 0, y \wedge z = 0\}$ for $y \in \Sigma$. Indeed, suppose $\bar{S}_1 \prec \bar{S}_2$ and $S_1 \cap \mathrm{Ann}(\Xi(S_2)) \neq \emptyset$. For a contradiction, suppose $S_1 \nsubseteq S_2$. Fix $y \in S_1 \setminus S_2$. Since $\bar{S}_1 \prec \bar{S}_2$, $\forall S_3 \in X \left(\bar{S}_1 \circ \bar{S}_3 \to \bar{S}_2 \circ \bar{S}_3\right)$, i.e. $\forall S_3 \in X \left(\bigwedge_{x \in S_1 \cup S_3} x \neq 0 \to \bigwedge_{x \in S_2 \cup S_3} x \neq 0\right)$. Taking $S_3 = \{y\}$, this means $\bigwedge_{x \in S_1 \cup \{y\}} x \neq 0 \to \bigwedge_{x \in S_2 \cup \{y\}} x \neq 0$. Since $\bigwedge_{x \in S_1} x \neq 0$ and $S_1 \cup \{y\} = S_1$, we get $\bigwedge_{x\in S_2\cup\{y\}} x \neq 0$. This means $\left(\bigwedge_{x\in S_2} x\right) \wedge y \neq 0$, i.e. $\Xi(S_2) \wedge y \neq 0$, so we have $y \notin \mathrm{Ann}(\Xi(S_2))$. y was an arbitrary member of $S_1 \setminus S_2$, so $(S_1 \setminus S_2) \cap \mathrm{Ann}(\Xi(S_2)) = \emptyset$. $S_1 \cap S_2 \subset S_2$ and $S_2 \cap \mathrm{Ann}(\Xi(S_2)) = \emptyset$, so $(S_1 \cap S_2) \cap \mathrm{Ann}(\Xi(S_2)) = \emptyset$. This implies $S_1 \cap \mathrm{Ann}(\Xi(S_2)) = \emptyset$, for a contradiction. Therefore $S_1 \subseteq S_2$.

Next, note that if $\bar{S_1} \prec \bar{S_2}$, then $\mathrm{Ann}(\Xi(S_2)) \subseteq \mathrm{Ann}(\Xi(S_1))$. Indeed, suppose $y \in \mathrm{Ann}(\Xi(S_2))$. Since $\forall S_3 \in X\left(\bigwedge_{x\in S_1\cup S_3} x \neq 0 \to \bigwedge_{x\in S_2\cup S_3} x \neq 0\right)$, taking $S_3 = \{y\}$ implies $\bigwedge_{x\in S_1\cup\{y\}} x \neq 0 \to \bigwedge_{x\in S_2\cup\{y\}} x \neq 0$. Since $\bigwedge_{x\in S_2\cup\{y\}} x \neq 0$ is false, so is $\bigwedge_{x\in S_1\cup\{y\}} x \neq 0$, and thus $\left(\bigwedge_{x\in S_1} x\right) \wedge y = 0$, so $y \in \mathrm{Ann}(\Xi(S_1))$, whence $\mathrm{Ann}(\Xi(S_2)) \subseteq \mathrm{Ann}(\Xi(S_1))$.

Remark $x \prec y \Leftrightarrow \alpha(x) \subseteq \alpha(y)$.

5 Points

Definition A <u>point</u> is any element $x \in X$ satisfying any of the following equivalent definitions:

1. $\forall y \in X\,(x \circ y \to x \prec y)$
2. $\forall y \in X\,(x \circ y \to \forall z \in X\,(x \circ z \to y \circ z))$
3. $\forall y, z \in X\,(x \circ y \wedge x \circ z \to y \circ z)$

Proposition 5.1 *If x_1 is a point and $x_2 \prec x_1$, then x_2 is a point.*

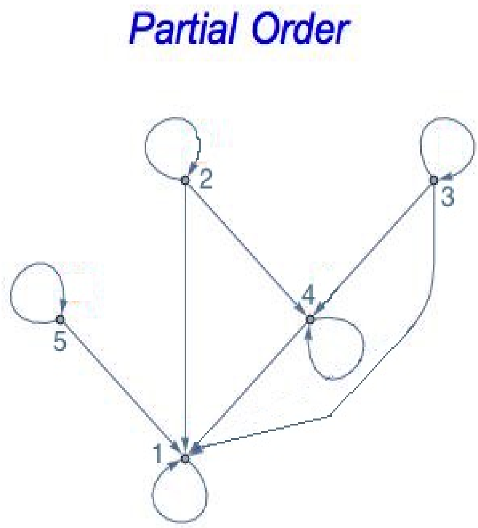

Figure 3: The partial order induced by the example in Figure 1.

Proof Suppose x_1 is a point and $x_2 \prec x_1$. Fix $y \in X$ such that $x_2 \circ y$. This is always possible because $x_2 \circ x_2$. Then since $x_2 \prec x_1$ and $x_2 \circ y$, we have $x_1 \circ y$. Since x_1 is a point, we have $x_1 \prec y$. Since $x_2 \prec x_1$, this implies $x_2 \prec y$. Hence $\forall y \in X\, (x_2 \circ y \rightarrow x_2 \prec y)$, which means x_2 is a point. ■

Proposition 5.2 *If $x_1, x_2 \in X$ are points and $x_1 \circ x_2$, then $x_1 = x_2$.*

Proof Suppose $x_1, x_2 \in X$ are points and $x_1 \circ x_2$. Then by definition of point, we have $x_1 \prec x_2$. Since $x_1 \circ x_2$, we have $x_2 \circ x_1$. Again by definition of point, we have $x_2 \prec x_1$. Since $\prec$ is a partial order, we then have that $x_1 = x_2$. ■

Remark If $x \in X$ is a point, then $\forall y, z \in X\, (x \prec y \wedge x \prec z \rightarrow y \circ z)$.

Proof Suppose x is a point and fix $y, z \in X$ such that $x \prec y$ and $x \prec z$. By Corollary 4.3, we have $x \circ y$ and $x \circ z$. Since x is a point, we get $y \circ z$. ■

<u>The converse is not true</u>. For example, consider the simple conjunctive space consisting of the three distinct elements x, y, and z, with conjunctive relation $\circ$ satisfying (exactly) $x \circ x$, $x \circ y$, $y \circ x$, $y \circ y$, $y \circ z$, $z \circ y$, and $z \circ z$. Then it is easily checked that x and z are points, but y is not. Further, it is easily checked that the induced partial order is described (exactly) by $x \prec x$, $x \prec y$, $y \prec y$, $z \prec y$, $z \prec z$. Thus we have $\forall p, q \in X\ (y \prec p \wedge y \prec q \rightarrow p \circ q)$ rather trivially, since the only critical choices of p and q are $p = q = y$, yielding the implication $y \prec y \wedge y \prec y \rightarrow y \circ y$ which is clearly satisfied.

Definition For $x \in X$, we define the sets $\iota(x) = \{y \in X; y \prec x\}$ and $\kappa(x) = \{y \in X; x \prec y\}$. Note that $\iota(x) \cap \kappa(x) = \{x\}$.

Remark x is a point $\Leftrightarrow \alpha(x) \subseteq \kappa(x)$.

Proposition 5.3 $\iota(a) \cap \iota(b) \neq \emptyset \Rightarrow a \circ b$

Proof Suppose $t \in \iota(a) \cap \iota(b)$. Since $t \in \iota(a)$, we have $t \prec a$, which implies $\forall y \in X\ (t \circ y \rightarrow a \circ y)$. In particular this means $t \circ b \rightarrow a \circ b$. Since $t \in \iota(b)$, and thus $t \prec b$, by Corollary 4.3 we have $t \circ b$. Thus it follows that $a \circ b$. ■

Proposition 5.4 *If p is a point, then $\iota(p) = \{p\}$.*

Proof Suppose p is a point and $b \in \iota(p)$. Then $b \prec p$. By Proposition 5.1, b is a point. Since $b \prec p$, then $b \circ p$, so by Proposition 5.2 we have that $b = p$. Thus $\iota(p) \subseteq \{p\}$. Clearly $p \in \iota(p)$, so $\{p\} \subseteq \iota(p)$, whence $\iota(p) = \{p\}$. ■

The converse is not true. In the example conjunctive space given above, the element y satisfies $\iota(y) = \{y\}$, yet y is not a point.

Proposition 5.5 $\iota(a) = \iota(b) \leftrightarrow a = b$

Proof Suppose $\iota(a) = \iota(b)$. Then

$\{y \in X; \forall z \in X\,(y \circ z \to a \circ z)\} = \{y \in X; \forall z \in X\,(y \circ z \to b \circ z)\}$. Since clearly $a \in \{y \in X; \forall z \in X\,(y \circ z \to a \circ z)\}$, we have

$a \in \{y \in X; \forall z \in X\,(y \circ z \to b \circ z)\}$, and thus $\forall z \in X\,(a \circ z \to b \circ z)$. Since clearly $b \in \{y \in X; \forall z \in X\,(y \circ z \to b \circ z)\}$, we have

$b \in \{y \in X; \forall z \in X\,(y \circ z \to a \circ z)\}$, and thus $\forall z \in X\,(b \circ z \to a \circ z)$, whence $\forall z \in X\,(a \circ z \leftrightarrow b \circ z)$, and thus $a = b$. ■

Proposition 5.6 *$x \prec y$ if and only if $\iota(x) \subseteq \iota(y)$.*

Proof ($\Rightarrow$) Suppose $x \prec y$. Fix $z \in \iota(x)$. Then $z \prec x$, and since $x \prec y$, we have $z \prec y$ and thus $z \in \iota(y)$, whence $\iota(x) \subseteq \iota(y)$. ($\Leftarrow$) Suppose $\iota(x) \subseteq \iota(y)$. Clearly $x \in \iota(x)$, hence $x \in \iota(y)$, thus $x \prec y$. ■

Remark A conjunctive space $(X, \circ)$ induces a category $C_{X,\circ}$. The objects of $C_{X,\circ}$ are the members of X. For a pair of (possibly identical) objects x and y, we construct a single arrow from x to y if $x \prec y$; otherwise there are no arrows from x to y. Then $C_{X,\circ}$ is a category. We then define a functor F from $C_{X,\circ}$ to the category **Set**. F takes each object $o \in C$ to $\iota(o)$ in **Set**, and takes each arrow $x \to y$ (i.e. $x \prec y$) to the arrow $\iota(x) \to \iota(y)$ (i.e. $\iota(x) \subseteq \iota(y)$).

Proposition 5.7 *x is a point if and only if $\forall y \in X\,(x \circ y \leftrightarrow x \prec y)$*

Proof $(\Rightarrow)$ Suppose x is a point. Then by definition of point, we have immediately $\forall y \in X\,(x \circ y \to x \prec y)$. Now fix $y \in X$ such that $x \prec y$. By Corollary 4.3, we have $x \circ y$, thus $\forall y \in X\,(x \circ y \leftarrow x \prec y)$, whence $\forall y \in X\,(x \circ y \leftrightarrow x \prec y)$.

$(\Leftarrow)$ Suppose $\forall y \in X\,(x \circ y \leftrightarrow x \prec y)$. It is immediate that

$\forall y \in X\,(x \circ y \to x \prec y)$, thus x is a point. ■

Proposition 5.8 *If* $y \in X$, *then* $\bigcap\limits_{x \in \alpha(y)} \alpha(x) \subseteq \alpha(y)$.

Proof Suppose $y \in X$. Since $y \in \alpha(y)$, we have $\bigcap\limits_{x \in \alpha(y)} \alpha(x) \subseteq \alpha(y)$. ■

Proposition 5.9 p *is a point* $\Leftrightarrow \alpha(p) = \bigcap\limits_{x \in \alpha(p)} \alpha(x)$

Proof $(\Rightarrow)$ Suppose p is a point. Fix $x \in \alpha(p)$. Then $p \circ x$, and since p is a point, we have $p \prec x$. This means $\alpha(p) \subseteq \alpha(x)$. Since x was an arbitrary member of $\alpha(p)$, we have $\alpha(p) \subseteq \bigcap\limits_{x \in \alpha(p)} \alpha(x)$.

Next, by Proposition 5.8, we have $\bigcap\limits_{x \in \alpha(p)} \alpha(x) \subseteq \alpha(p)$. We saw earlier that $\alpha(p) \subseteq \bigcap\limits_{x \in \alpha(p)} \alpha(x)$, so we have $\alpha(p) = \bigcap\limits_{x \in \alpha(p)} \alpha(x)$.

$(\Leftarrow)$ Suppose $\alpha(p) = \bigcap\limits_{x \in \alpha(p)} \alpha(x)$. Fix y such that $p \circ y$. This means $y \in \alpha(p)$, so $\bigcap\limits_{x \in \alpha(p)} \alpha(x) \subseteq \alpha(y)$. Since $\alpha(p) = \bigcap\limits_{x \in \alpha(p)} \alpha(x)$, we get $\alpha(p) \subseteq \alpha(y)$, which means $p \prec y$. Thus p is a point. ■

Remark In Example 1, every $x \in X$ is a point.

In Example 2, x is a point if and only if it is a subset of any set which it meets.

In Example 3, for any point N, if $\bigcap_{n\in N} n \neq \emptyset$ is nonempty, it is a singleton set. Indeed, suppose $\{a,b\} \subseteq \bigcap_{n\in N} n \neq \emptyset$ for some $a \neq b$. Take $N_1 = \{\{a\}\}$. Since N is a point, we have

$\forall N_1 \left(\forall n \in N \forall n_1 \in N_1 \left(n \cap n_1 \neq \emptyset\right) \to \forall n_1 \in N_1 \exists n \in N \left(n \subseteq n_1\right)\right)$, so in particular $\forall n \in N \forall n_1 \in \{\{a\}\} \left(n \cap n_1 \neq \emptyset\right) \to \forall n_1 \in \{\{a\}\} \exists n \in N \left(n \subseteq n_1\right)$, or $\forall n \in N \left(n \cap \{a\} \neq \emptyset\right) \to \exists n \in N \left(n \subseteq \{a\}\right)$. Since $\forall n \in N \left(n \cap \{a\} \neq \emptyset\right)$, we get $\exists n \in N \left(n \subseteq \{a\}\right)$. Let n_0 be such an n. Then $n_0 \subseteq \{a\}$. Since $\{a,b\} \subseteq \bigcap_{n\in N}$ and $\bigcap_{n\in N} n \subseteq n_0 \subseteq \{a\}$, it follows that $\{a,b\} \subseteq \{a\}$, which is absurd. Thus $\bigcap_{n\in N} n$ cannot contain more than one distinct member. Since $\bigcap_{n\in N} n \neq \emptyset$, $\bigcap_{n\in N} n$ is a singleton set.

6 Nests

In the following all chains are members of $\Phi(X, \circ)$.

Definition We define a relation $\square$ on chains satisfying $C_1 \square C_2$ if and only if $\forall x_1 \in C_1 \forall x_2 \in C_2 \left(x_1 \circ x_2\right)$. This relation is symmetric and reflexive.

Remark $C_1 \square C_2$ if and only if

$$C_1 \subseteq \bigcap_{x\in C_2} \alpha(x), \tag{6.1}$$

or equivalently

$$C_2 \subseteq \bigcap_{x\in C_1} \alpha(x). \tag{6.2}$$

Definition We define a relation on chains satisfying $C_1 \sim C_2$ if and only $\forall C_3 \left(C_1 \Box C_3 \leftrightarrow C_2 \Box C_3 \right).$

Remark $C_1 \sim C_2$ if and only if

$$\bigcap_{x \in C_1} \alpha(x) = \bigcap_{x \in C_2} \alpha(x). \tag{6.3}$$

Proposition 6.1 $\sim$ *is an equivalence relation on the set of chains.*

Proof (1) Let C_1 be a chain. Since

$(\forall x_1 \in C_1 \forall x_3 \in C_3 \left(x_1 \circ x_3 \right)) \leftrightarrow (\forall x_2 \in C_1 \forall x_3 \in C_3 \left(x_2 \circ x_3 \right))$ for every chain C_3, we have $C_1 \sim C_1$.

(2) Let C_1 and C_2 be chains such that $C_1 \sim C_2$. Then

$(\forall x_1 \in C_1 \forall x_3 \in C_3 \left(x_1 \circ x_3 \right)) \leftrightarrow (\forall x_2 \in C_2 \forall x_3 \in C_3 \left(x_2 \circ x_3 \right))$ for every chain C_3. This implies

$(\forall x_2 \in C_2 \forall x_3 \in C_3 \left(x_2 \circ x_3 \right)) \leftrightarrow (\forall x_1 \in C_1 \forall x_3 \in C_3 \left(x_1 \circ x_3 \right))$ for every chain C_3, which means $C_2 \sim C_1$.

(3) Suppose $C_1 \sim C_2$ and $C_2 \sim C_3$. Then $(\forall x_1 \in C_1 \forall x \in C \left(x_1 \circ x \right)) \leftrightarrow (\forall x_2 \in C_2 \forall x \in C \left(x_2 \circ x \right))$ and

$(\forall x_2 \in C_2 \forall x \in C \left(x_2 \circ x \right)) \leftrightarrow (\forall x_3 \in C_3 \forall x \in C \left(x_3 \circ x \right))$ for every chain C. Stringing these together we get

$(\forall x_1 \in C_1 \forall x \in C \left(x_1 \circ x \right)) \leftrightarrow (\forall x_3 \in C_3 \forall x \in C \left(x_3 \circ x \right))$ for every chain C. This means $C_1 \sim C_3$.$\blacksquare$

Definition By <u>nest</u> we mean an equivalence class of chains.

Consider the collection of nests $Y = \{\bar{C}; C \in \Phi(X, \circ)\}$. For $\bar{C}_1, \bar{C}_2 \in Y$, define the relation $\bar{C}_1 \bar{\circ} \bar{C}_2$ if and only if $C_1 \square C_2$. Then $(Y, \bar{\circ})$ is a conjunctive space.

7 Defining a Conjunctive Relation On Nests

Notation We write $\bar{X}$ to denote the set of nests in the conjunctive space $(X, \circ)$ induced by partial order $\prec$.

Notation By $\bar{C}$ is meant the nest containing the chain C.

Definition We define a relation on nests satisfying $\bar{C}_1 \bar{\circ} \bar{C}_2$ if $\forall x_1 \in C_1 \forall x_2 \in C_2 \, (x_1 \circ x_2)$.

Proposition 7.1 *$\bar{\circ}$ is a conjunctive relation on $\bar{X}$, and therefore $\left(\bar{X}, \bar{\circ}\right)$ is a conjunctive space.*

Proof (1) Suppose $\bar{C}_1$ is a nest. Then since C_1 is a chain, $\forall x_1 \in C_1 \forall x_2 \in C_1 \, (x_1 \circ x_2)$, which means $\bar{C}_1 \bar{\circ} \bar{C}_1$. (2) Suppose $\bar{C}_1$ and $\bar{C}_2$ are nests such that $\bar{C}_1 \bar{\circ} \bar{C}_2$. Then $\forall x_1 \in C_1 \forall x_2 \in C_2 \, (x_1 \circ x_2)$, which means $\forall x_2 \in C_2 \forall x_1 \in C_1 \, (x_2 \circ x_1)$, hence $\bar{C}_2 \bar{\circ} \bar{C}_1$. (3) Suppose $\bar{C}_1$ and $\bar{C}_2$ are nests such that $\bar{C}_1 \bar{\circ} \bar{C}_3 \leftrightarrow \bar{C}_2 \bar{\circ} \bar{C}_3$ for every nest $\bar{C}_3$. Then $\forall x_1 \in C_1 \forall x_3 \in C_3 \, (x_1 \circ x_3) \leftrightarrow \forall x_2 \in C_2 \forall x_3 \in C_3 \, (x_2 \circ x_3)$ for every nest $\bar{C}_3$, hence $\bar{C}_1 = \bar{C}_2$.$\blacksquare$

Remark Under the induced partial order $\bar{\prec}$, $\bar{C}_1 \bar{\prec} \bar{C}_2$ if and only if

$\forall C_3\left(\left(\forall x\in C_1\forall y\in C_3\left(x\circ y\right)\right)\rightarrow\left(\forall z\in C_2\forall w\in C_3\left(z\circ w\right)\right)\right)$. Alternatively, Under the induced partial order $\bar{\prec}$, $\bar{C}_1\bar{\prec}\bar{C}_2$ if and only if

$$\forall C_1\in\bar{C}_1\forall C_2\in\bar{C}_2\left(\bigcap_{x\in C_1}\alpha(x)\subseteq\bigcap_{x\in C_2}\alpha(x)\right). \tag{7.1}$$

Equivalently, $\bar{C}_1\bar{\prec}\bar{C}_2$ if and only if

$$\bigcup_{C\in\bar{C}_1}\bigcap_{x\in C}\alpha(x)\subseteq\bigcap_{C\in\bar{C}_2}\bigcap_{x\in C}\alpha(x). \tag{7.2}$$

This suggests an alternative definition of a nest-point: $\bar{P}\in\bar{X}$ is a nest-point if and only if

$$\forall\bar{Q}\in\bar{X}\left(\bar{P}\bar{\circ}\bar{Q}\Rightarrow\bigcup_{C\in\bar{P}}\bigcap_{x\in C}\alpha(x)\subseteq\bigcap_{C\in\bar{Q}}\bigcap_{x\in C}\alpha(x)\right). \tag{7.3}$$

Remark The conjunctive space of nests $\left(\bar{X},\bar{\circ}\right)$ features a definition of a nest that is a point. From the definition, a nest $\bar{C}$ is a point, i.e. a <u>nest-point</u> if and only if it satisfies $\forall\bar{C}_1,\bar{C}_2\in\bar{X}\left(\bar{C}\bar{\circ}\bar{C}_1\wedge\bar{C}\bar{\circ}\bar{C}_2\rightarrow\bar{C}_1\bar{\circ}\bar{C}_2\right)$. Equivalently, for every choice of chains $C_1,C_2\in\Phi\left(X,\circ\right)$, $\forall x_1\in C\forall x_2\in C_1\left(x_1\circ x_2\right)\wedge\forall y_1\in C\forall y_2\in C_2\left(y_1\circ y_2\right)$ implies $\forall z_1\in C_1\forall z_2\in C_2\left(z_1\circ z_2\right)$.

8 Rational Conjunctive Space

We now consider an interesting example, namely the rational conjunctive space referred to in Section 2. Recall the definition: Let $X = \{\{p \in \mathbb{Q}; \lambda(x) \leqslant p \leqslant \rho(x)\}; \lambda(x), \rho(x) \in \mathbb{Q}, \lambda(x) \leqslant \rho(x)\}$, and for any $x_1, x_2 \in X$, define the relation $x_1 \circ x_2$ if and only if $x_1 \cap x_2 \neq \emptyset$, i.e. $\max(\lambda(x_1), \lambda(x_2)) \leqslant \min(\rho(x_1), \rho(x_2))$. If this inequality holds, then in fact $x_1 \cap x_2 = \{p \in \mathbb{Q}; \max(\lambda(x_1), \lambda(x_2)) \leqslant p \leqslant \min(\rho(x_1), \rho(x_2))\}$ and $x_1 \cap x_2 \in X$. The partial order $\prec$ satisfies $x_1 \prec x_2$ if and only if $x_1 \subseteq x_2$. A chain is any nested (in the sense of set inclusion) collection of members of X. Recall that by definition,

$$C_1 \sim C_2 \Leftrightarrow$$
$$\forall C_3 \in \Phi(X, \circ) \left(\forall x_1 \in C_1 \forall x_3 \in C_3 \left(x_1 \cap x_3 \neq \emptyset\right) \leftrightarrow \forall x_2 \in C_2 \forall x_3 \in C_3 \left(x_2 \cap x_3 \neq \emptyset\right)\right). \tag{8.1}$$

Definition For any chains C, $\sigma(C) = \left(\sup\limits_{x \in C} \lambda(x), \inf\limits_{x \in C} \rho(x)\right)$.

Proposition 8.1 *For any chains C_1 and C_2 if $C_1 \sim C_2$, then $\sigma(C_1) = \sigma(C_2)$.*

Proof Suppose $C_1 \sim C_2$. Let $L_1 = \sup\limits_{x \in C_1} \lambda(x)$ and $R_1 = \inf\limits_{x \in C_1} \rho(x)$, and let $L_2 = \sup\limits_{y \in C_2} \lambda(y)$. Suppose for a contradiction that $L_1 \neq L_2$. Without loss of generality we can assume $L_1 < L_2$. Fix $\epsilon > 0$ such that $L_1 < L_2 - \epsilon$. We can find some $y \in C_2$ such that $L_2 - \epsilon < \lambda(y)$, thus $y \subset (L_2 - \epsilon, \infty)$. We can find $q_1, q_2 \in \mathbb{Q}$ such that $L_1 \leqslant q_1 \leqslant q_2 \leqslant L_2 - \epsilon$. Now take $C_3 \in \Phi(X, \circ)$ to be the single-member chain $\{\{p \in \mathbb{Q}; q_1 \leqslant p \leqslant q_2\}\}$. Clearly $\{p \in \mathbb{Q}; q_1 \leqslant p \leqslant q_2\} \subseteq [q_1, q_2] \subseteq [L_1, L_2 - \epsilon]$, and since y does not meet $[L_1, L_2 - \epsilon]$, we have that $y \cap \{p \in \mathbb{Q}; q_1 \leqslant p \leqslant q_2\} = \emptyset$. However, $y \cap x \neq \emptyset$ for every $x \in C_1$, so it is not true that $\forall x_1 \in C_1 \forall x_3 \in C_3 \, (x_1 \cap x_3 \neq \emptyset) \leftrightarrow \forall x_2 \in C_2 \forall x_3 \in C_3 \, (x_2 \cap x_3 \neq \emptyset)$ for a contradiction. Thus $L_1 = L_2$, i.e. $\sup\limits_{x \in C_1} \lambda(x) = \sup\limits_{y \in C_2} \lambda(y)$. A similar argument gives $\inf\limits_{x \in C_1} \rho(x) = \inf\limits_{y \in C_2} \rho(y)$, thus $\sigma(C_1) = \sigma(C_2)$. ■

Proposition 8.2 $C_1 \sim C_2 \Rightarrow \forall x \in X \, (\forall y \in C_1 \, (x \subseteq y) \leftrightarrow \forall z \in C_2 \, (x \subseteq z))$.

Proof Suppose $C_1 \sim C_2$. Suppose $x \in X$ such that $\forall y \in C_1 \, (x \subseteq y)$. For a contradiction suppose $x \nsubseteq q$ for some $q \in C_2$. Fix nonempty p such that $p \subseteq x \backslash q$. Take $C_3 = \{p\}$. Then since $C_1 \sim C_2$, $\forall x_1 \in C_1 \, (x_1 \cap p \neq \emptyset) \leftrightarrow \forall x_2 \in C_2 \, (x_2 \cap p \neq \emptyset)$. Clearly $\forall x_1 \in C_1 \, (x \subseteq x_1)$, and $p \subseteq x$, so $\forall x_1 \in C_1 \, (p \subseteq x_1)$, thus $\forall x_1 \in C_1 \, (x_1 \cap p \neq \emptyset)$, whence $\forall x_2 \in C_2 \, (x_2 \cap p \neq \emptyset)$. Recall $q \in C_2$, so we get $q \cap p \neq \emptyset$. But $q \cap p \subseteq q \cap (x \setminus q) = \emptyset$, so $q \cap p = \emptyset$ for a contradiction. Therefore $x \subseteq q$. Thus $\forall x \in X \, (\forall y \in C_1 \, (x \subseteq y) \to \forall z \in C_2 \, (x \subseteq z))$. A similar argument gives $\forall x \in X \, (\forall z \in C_2 \, (x \subseteq z)) \to \forall x \in X \, (\forall y \in C_1 \, (x \subseteq y))$, hence $\forall x \in X \, (\forall y \in C_1 \, (x \subseteq y) \leftrightarrow \forall z \in C_2 \, (x \subseteq z))$. ■

Remark Each nest that is a point corresponds to a real number.

9 Contact

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10 Bibliography

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